Announcements

1) Exam next week
   covers 11.4, 11.5, 11.8 – 11.10
   10.1 – 10.4 (no arclength or area in parametric curve)

2) Lab due tomorrow

3) Practice Problems, exams up tomorrow

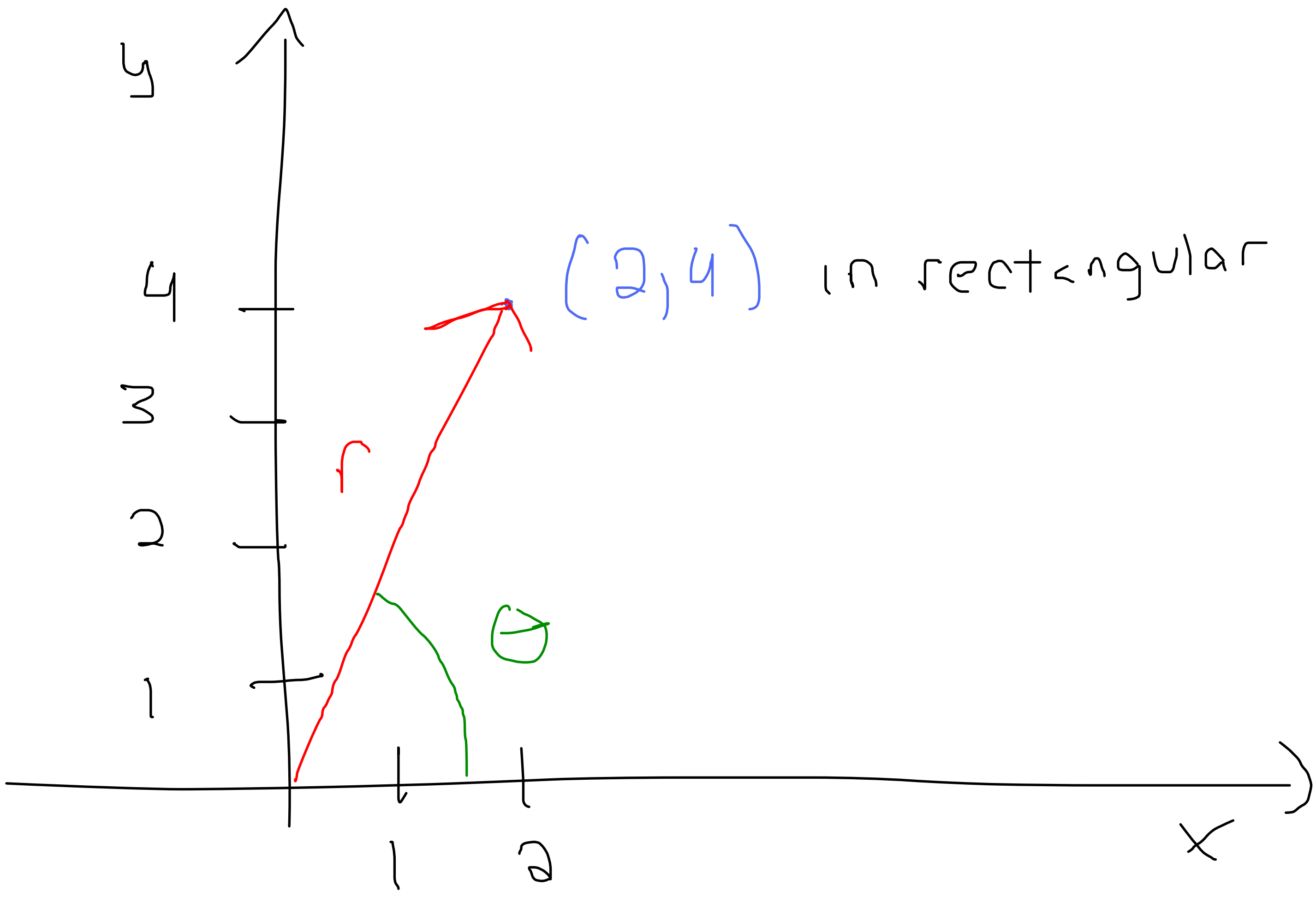
Polar Coordinates

Section 10.3

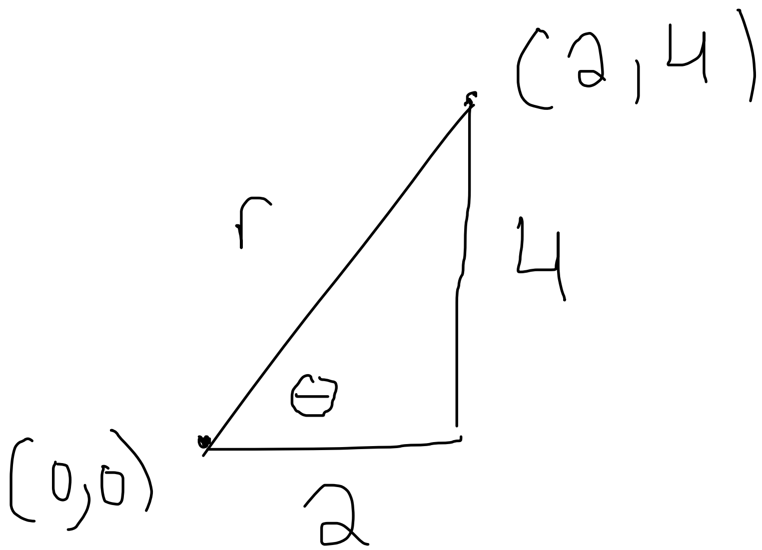
Magnitude plus direction

Instead of considering a point in 2 dimensions using (x, y) coordinates, change to a different coordinate system

Picture



The polar coordinates of (2, 4) will be (r, θ) where...



$$r^2 = 2^2 + 4^2 = 4 + 16 = 20,$$

so $\boxed{r = 2\sqrt{5}}$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{4}{2} = 2$$

$$\boxed{\theta = \arctan(2)}$$

Conversions

Given (x, y) in rectangular (Cartesian) coordinates, the conversion to polar coordinates is

$$(x, y) \mapsto \left(\overbrace{\sqrt{x^2 + y^2}}^{r}, \overbrace{\arctan\left(\frac{y}{x}\right)}^{\theta} \right)$$

except when $x = 0$.

If $x = 0$, then

$$(0, y) \mapsto (y, 0) \text{ if } y \geq 0$$

or

$$(0, y) \mapsto (-y, 0) \text{ if } y < 0$$

Given (r,θ) in polar coordinates, the rectangular (Cartesian) coordinates are given by

$$(r,\theta) \mapsto (\underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y})$$

Warning: (non uniqueness)

If (r, θ) is given in polar coordinates, then the rectangular coordinates of (r, θ) are identical to those of $(r, \theta + 2n\pi)$ where n is an integer:

$n = 0, 1, 2, 3, \ldots$
or $-1, -2, -3, \ldots$

Example 1: (polar to Cartesian)

Given $(r, \theta) = \left(12, \frac{\pi}{6}\right)$, compute the Cartesian coordinates.

$$x = r\cos(\theta) = 12\cos\left(\frac{\pi}{6}\right) = \frac{12\sqrt{3}}{2}$$
$$= 6\sqrt{3}$$

$$y = r\sin(\theta) = 12\sin\left(\frac{\pi}{6}\right) = 12 \cdot \frac{1}{2}$$
$$= 6$$

We get $\boxed{(6\sqrt{3}, 6)}$

Example 2: (Cartesian to Polar)

Given $(x, y) = (52, 17)$

find two different representations in polar coordinates.

$$r = \sqrt{x^2+y^2} = \sqrt{289+2704}$$

$$= \sqrt{2993}$$

$$\theta = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{17}{52}\right)$$

We get $\boxed{\left(\sqrt{2993}, \arctan\left(\frac{17}{52}\right)\right)}$

Also $\boxed{\left(\sqrt{2993},\ \arctan\left(\frac{17}{52}\right)+2\pi\right)}$

Convention: (negative radius)

Observe that points of the form

$$\left(\sqrt{x^2+y^2},\ \arctan\left(\frac{y}{x}\right)\right)$$

are only in the first and fourth quadrants. How to get the second and third?

If you're given (x, y) in either the second or third quadrants, its polar form can be found by

$$(x, y) \mapsto \left(\sqrt{x^2+y^2}, \arctan\left(\frac{y}{x}\right)+\pi\right)$$

Convention: The point $(-r, \theta)$ in polar is defined as $(r, \theta+\pi)$ (180° away from (r, θ))

Example 3: Plot

$$\left(-2.5, -\frac{5\pi}{3}\right),$$

given in polar coordinates.

Making our lives easier first.

$$\left(-2.5, -\frac{5\pi}{3}\right) = \left(2.5, -\frac{5\pi}{3} + \pi\right)$$

$\theta = -\frac{5\pi}{3} + \frac{3\pi}{3}$

$$= \left(2.5, -\frac{2\pi}{3}\right)$$

$$= \left(2.5, -\frac{2\pi}{3} + 2\pi\right)$$

$\theta = -\frac{2\pi}{3} + \frac{6\pi}{3}$

$$= \left(2.5, \frac{4\pi}{3}\right)$$

Convert:

$$x = 2.5\cos\left(\frac{4\pi}{3}\right) = 2.5\left(-\frac{1}{2}\right)$$
$$= -\frac{5}{4}$$

$$y = 2.5\sin\left(\frac{4\pi}{3}\right) = 2.5\left(-\frac{\sqrt{3}}{2}\right)$$
$$= -\frac{5\sqrt{3}}{4}$$

So $\boxed{\left(-\frac{5}{4}, -\frac{5\sqrt{3}}{4}\right)}$

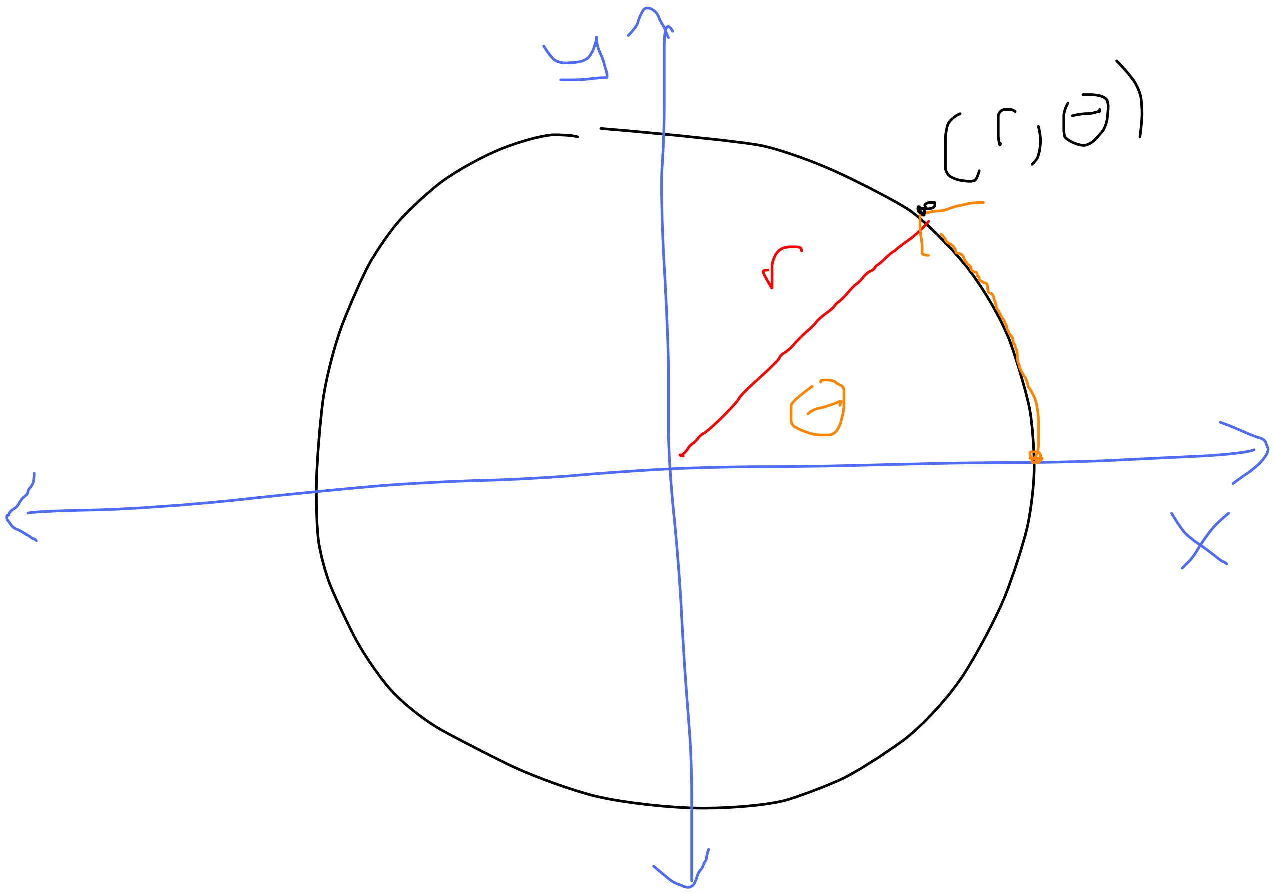
Graphing with Polar Coordinates

How do you plot a point (r, θ) in polar coordinates?

If $r \geq 0$, no problem.

If $r < 0$, replace (r, θ) with $(-r, \theta + \pi)$, follow subsequent directions!

First, draw a circle about $(0,0)$ of radius r,



Second, starting at the point where the circle touches the positive x-axis, move counterclockwise θ radians

Example 4: (just r)

$$r = 5$$

Switch to Cartesian:

$$r = \sqrt{x^2+y^2}, \quad \text{so}$$

$$\sqrt{x^2+y^2} = 5 \quad \text{square both sides}$$

$$x^2+y^2 = 5^2 = 25$$

Circle of radius 5, center (0,0)

Example 5: (just θ)

$$\theta = \frac{\pi}{7}$$

Switch back to cartesian

$$\theta = \arctan\left(\frac{y}{x}\right), \text{ so}$$

$$\frac{\pi}{7} = \arctan\left(\frac{y}{x}\right)$$ take tangent of both sides

$$\frac{y}{x} = \tan\left(\frac{\pi}{7}\right)$$ (multiply both sides by x)

$$y = \tan\left(\frac{\pi}{7}\right)x$$

$\boxed{\text{Line through } (0,0) \text{ with slope } \tan\left(\frac{\pi}{7}\right)}$

Example 6. $r = \cos\theta$

Switch back to Cartesian:

$$r = \sqrt{x^2+y^2}$$

$x = r\cos(\theta)$, so $\cos(\theta) = \frac{x}{r}$

$$= \frac{x}{\sqrt{x^2+y^2}}$$

Substitute

$$\sqrt{x^2+y^2} = \frac{x}{\sqrt{x^2+y^2}}$$ (multiply both sides by $\sqrt{x^2+y^2}$)

$$x^2+y^2 = x$$

$x^2 + y^2 = x$, so subtracting x from both sides,

$$x^2 - x + y^2 = 0$$

Complete the square for x!

$$x^2 - x = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4}$$

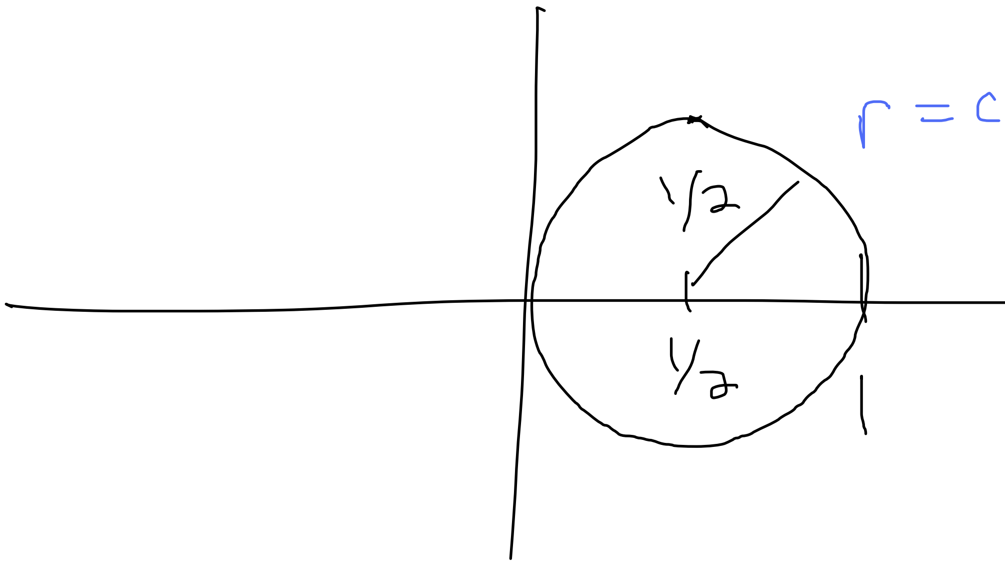
Check!

So we get

$\left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + y^2 = 0$ and

$$\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$$

Circle with center $(1/2, 0)$
and radius $1/2$



$r = \cos\theta$

Area and Polar Curves

(Section 10.4)

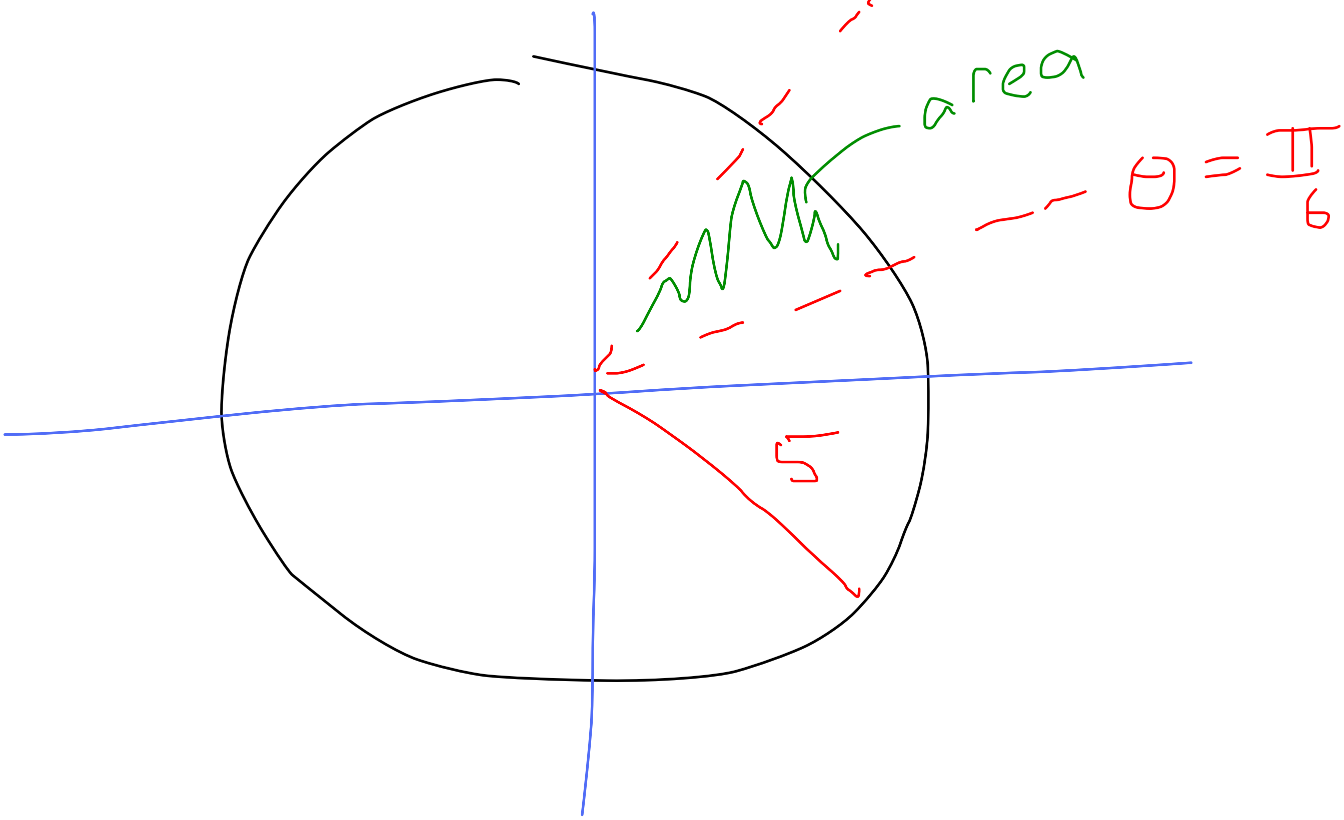
$$\int_{\pi/6}^{\pi/3} 5 \, d\theta$$ – how to interpret?

In polar: this is the area inside $r = 5$ from $\theta = \frac{\pi}{6}$ to $\theta = \frac{\pi}{3}$

circle

rays

Picture



The area of a sector –

in general, the area of a sector determined by angle θ in a circle of radius r is

$$\boxed{A = \frac{1}{2} r^2 \theta}$$

Formula: In polar coordinates, we use sectors instead of rectangles for integrals.

So: The area inside $r = f(\theta)$ from $\theta = \theta_0$ to $\theta = \theta_1$ is

$$A = \frac{1}{2} \int_{\theta_0}^{\theta_1} (f(\theta))^2 \, d\theta$$

Don't forget the square!

Example 7 Area in one loop of $r = \sin(4\theta)$

Need to graph to find the θ values for one loop!

Loop. Start where $r = 0$.

End at the next θ value where $r = 0$.

We want to find

where $r = \sin(4\theta) = 0$

This happens when

$4\theta = 0, \pi, 2\pi, 3\pi$, etc

Dividing by 4,

$\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$, etc

Pick two consecutive values!

$\theta = 0, \frac{\pi}{4}$

Area is then

$$\frac{1}{2}\int_0^{\pi/4} \left(\sin(4\theta)\right)^2 d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1-\cos(8\theta)}{2}\, d\theta$$

$$\left(\sin^2(x) = \frac{1-\cos(2x)}{2} \right.$$

$$\left. x = 4\theta \right)$$

$$= \frac{1}{4}\int_0^{\pi/4} \left(1-\cos(8\theta)\right) d\theta$$

$$= \frac{1}{4}\left(\theta - \frac{\sin(8\theta)}{8}\right)\Bigg|_0^{\pi/4}$$

$$\frac{1}{4}\left(\theta - \frac{\sin(8\theta)}{8}\right)\Bigg|_0^{\pi/4}$$

$$= \frac{1}{4}\left(\left(\frac{\pi}{4} - \frac{\sin\left(8\frac{\pi}{4}\right)}{8}\right) - \underbrace{\left(0 - \frac{\sin(0)}{8}\right)}_{=0}\right)$$

$$= \frac{1}{4}\left(\frac{\pi}{4} - \underbrace{\frac{\sin(2\pi)}{8}}_{=0}\right)$$

$$= \boxed{\frac{\pi}{16}} \leftarrow \text{Area}$$

Example 8:

Area inside $r = 6\cos\theta$ but

Outside $r = 2+2\cos\theta$

Must draw!

$r = \cos\theta$ – circle, center $(1/2, 0)$

radius $1/2$

Multiplying by 6 expands

this to center $3 = 6 \cdot 1/2$

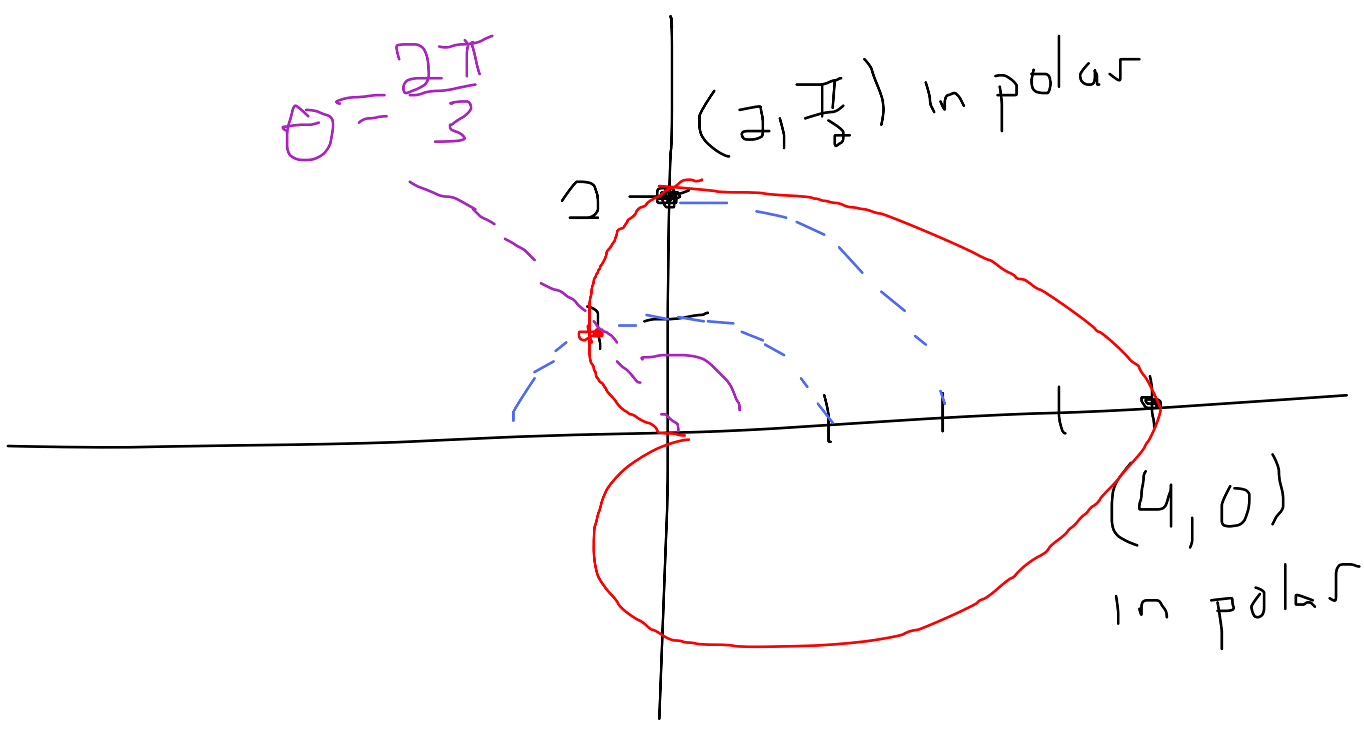
and radius $3 = 6 \cdot 1/2$

$$r = 2 + 2\cos\theta$$

$$\frac{dr}{d\theta} = -2\sin\theta$$

decreasing for $0 < \theta < \pi$

Plot $r(0) = 2 + 2\cos(0) = 4$



$$r\left(\frac{\pi}{2}\right) = 2 + 2\cos\left(\frac{\pi}{2}\right) = 2$$

polar point $\left(2, \frac{\pi}{2}\right)$

$$r\left(\frac{2\pi}{3}\right) = 2 + 2\cos\left(\frac{2\pi}{3}\right)$$

$$= 2 + 2\left(-\frac{1}{2}\right)$$

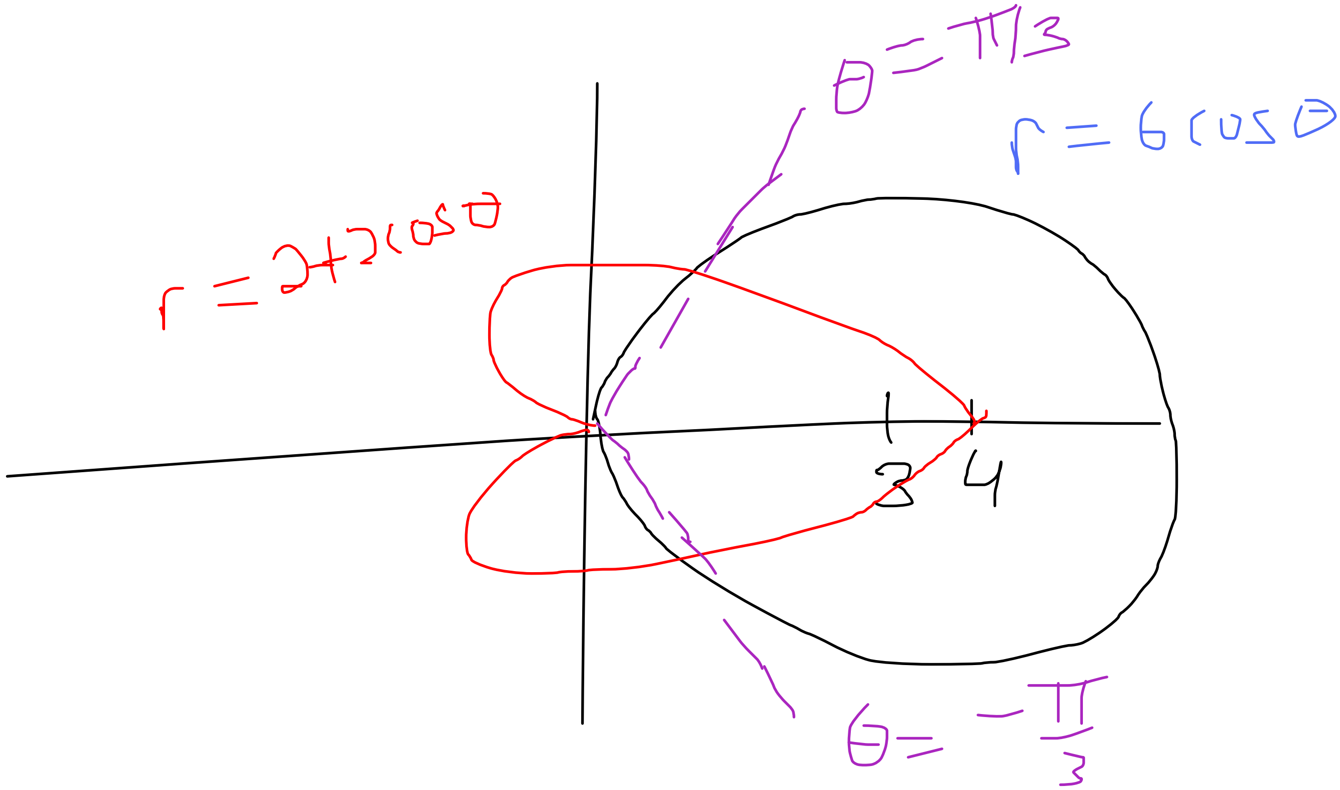
$$= 1$$

polar point $\left(1, \frac{2\pi}{3}\right)$

$$r(\pi) = 2 + 2\cos(\pi)$$

$$= 2 - 2 = 0$$

polar point $(0, \pi)$



Area inside $r = 6\cos\theta$,
outside $r = 2+2\cos\theta$

Find intersection

$$6\cos\theta = 2+2\cos\theta$$

$$4\cos\theta = 2,$$

$$\cos\theta = 1/2 \quad \theta = \frac{\pi}{3}, -\frac{\pi}{3}$$

Since picture

is symmetric

about x-axis,

go from $\theta = 0$ to $\theta = \frac{\pi}{3}$,

double the Areal

$$A = 2 \cdot \frac{1}{2} \int_0^{\pi/3} (6\cos\theta)^2 - (2+2\cos\theta)^2 \, d\theta$$

Solve!